data the shape factor for spheroidal material is unity.

NOTATION

A = coefficient, lb.$_f$
D = tube diameter, ft.
D_p = particle diameter, ft.
g_c = conversion factor, (lb.$_m$/lb.$_f$) (ft./sec.^{2})
k_1 = coefficient, lb.$_f$/sq. ft.
k_2 = coefficient, dimensionless
ln = logarithm to base e
L/x = particle edge to thickness ratio, dimensionless
S/S_o = platelike particle surface area/equiaxial particle surface area, dimensionless
V = mean velocity, ft./sec.

Greek Letters

η = coefficient of rigidity, lb.$_m$/ft. sec.
μ = viscosity of suspending medium, lb.$_m$/ft. sec.
σ = logarithmic standard deviation, dimensionless
τ_y = yield stress, lb.$_f$/sq. ft.
τ_w = wall shear stress, lb.$_f$/sq. ft.
ϕ = volume fraction solids, dimensionless
ψ = shape factor, dimensionless

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Laminar Boundary Layer Flow and Heat Transfer Past a Flat Plate For a Liquid of Variable Viscosity

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A solution for the problem of incompressible laminar boundary-layer flow and heat transfer with variable viscosity is presented. Because of the variation of viscosity with temperature the velocity and temperature fields interact mutually. This necessitates the simultaneous solution of the momentum and energy equations. The analysis is carried out for the case where heating begins at the leading edge of the plate. The results show the effect of the important variable property parameters on the friction factor and the heat transfer coefficient. These parameters are seen to be the temperature difference between wall and free stream, the viscosity temperature variation law, and the Prandtl number at the wall. The results are applicable to liquids.

The laminar boundary-layer equations for incompressible flow and heat transfer past a flat plate maintained at a constant temperature have been solved exactly for constant properties (*1*). In addition Eckert (*2*) has given a useful approximate solution for flow with constant properties, which accounts for the effect of an unheated starting length on heat transfer. When heating starts at the leading edge, the Eckert solution reduces to a form which is in agreement with the exact solution. These relations are useful in many cases, but there is doubt concerning their validity in the case of

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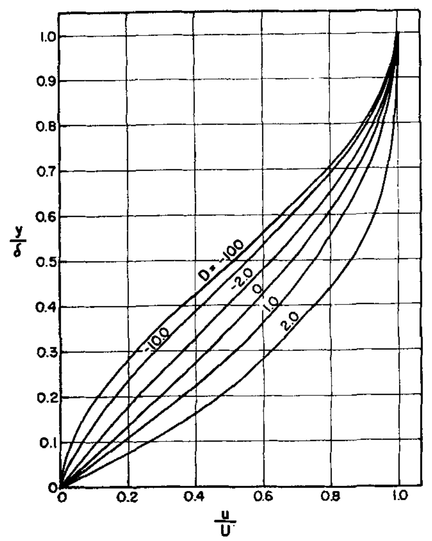

Fig. 1. The effect of variable viscosity on the velocity profile.

high-temperature differences or for fluids having viscosities that are strongly temperature dependent. Several attempts have been made to take into account variable fluid properties in different flow situations. Eckert (3) suggests a reference temperature to evaluate fluid properties for laminar flow past a flat plate when the Prandtl number is constant (gases). For laminar flow in a tube Sieder and Tate (4) recommend a correction factor of $(\mu_b/\mu_w)^n$ to account for nonisothermal effects on heat transfer and friction coefficients. The purpose of this paper is to clarify the effects of variable viscosity on heat and momentum transfer for incompressible laminar flow of a liquid past a flat plate.

In forced flow the velocity and temperature fields do not interact mutually when fluid properties are constant. However when the viscosity varies the fields do interact, and thus the situation becomes much more complex. In this analysis the viscosity will be assumed to be an arbitrary function of temperature, while the thermal conductivity will be assumed to be constant. This is the case with most liquids.

ANALYSIS

Consider the laminar boundary-layer equations of momentum and energy written for incompressible flow with constant heat capacity and thermal conductivity:

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) \quad (1)$$

$$C_p\rho\left(u\frac{\partial t}{\partial x}+v\frac{\partial t}{\partial y}\right)=k\frac{\partial^2 t}{\partial y^2} \quad (2)$$

By integration with respect to y between the wall and the edge of the momentum and thermal boundary layers respectively these equations are put into their well-known integral forms:

$$\frac{d}{dx}\int_0^{\delta_1}\frac{u}{U}\left(1-\frac{u}{U}\right)dy=\left.\frac{\nu_w}{U^2}\frac{\partial u}{\partial y}\right)_w \quad (3)$$

$$\frac{d}{dx}\int_0^{\delta'}\frac{u}{U}\left(1-\frac{\theta}{\bar{\theta}}\right)dy=\left.\frac{\alpha}{U\bar{\theta}}\frac{\partial\theta}{\partial y}\right)_w \quad (4)$$

Here $\bar{\theta}=t_\infty-t_w$ is constant. If the velocity and temperature profiles are known, the heat and momentum transfer problems can be solved by Equations (3) and (4). Eckert (2) has successfully applied this method to make heat and momentum transfer calculations for incompressible laminar flow past a flat plate of a fluid with constant properties. The general procedure here will follow that of Eckert with one principal exception. The estimate of the velocity profile will be made on the assumption that the fluid viscosity is an arbitrary function of temperature. The final results reduce to those of Eckert in the limiting case of constant viscosity.

It is assumed that the velocity profile can be approximated by a third-degree polynomial:

$$u=C_1+C_2y+C_3y^2+C_4y^3 \quad (5)$$

Here C_1 to C_4 may be functions of x. This profile must satisfy the conditions

(a) $u=0$ at $y=0$

(b) $u=U$ at $y=\delta$

(c) $\frac{\partial u}{\partial y}=0$ at $y=\delta$

And finally, applying Equation (1) at $y=0$, one gets

(d) $\mu\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial y}\frac{\partial \mu}{\partial y}=0$ at $y=0$

Using these conditions to evaluate C_1 to C_4 one obtains

$$\frac{u}{U}=\frac{\frac{3}{2}\frac{y}{\delta}-\frac{3}{4}A\left(\frac{y}{\delta}\right)^2-\frac{1}{2}(1-A)\left(\frac{y}{\delta}\right)^3}{1-0.25A} \quad (6)$$

where

$$A=\left.\frac{\delta}{\mu_w}\frac{\partial\mu}{\partial y}\right)_w \quad (7)$$

Since μ is an arbitrary function of temperature

$$\left.\frac{\partial\mu}{\partial y}\right)_w=\left.\frac{d\mu}{dt}\right)_w\left.\frac{\partial t}{\partial y}\right)_w=\mu'(t_w)\left.\frac{\partial t}{\partial y}\right)_w \quad (8)$$

Thus

$$A=\frac{\delta\,\mu'(t_w)}{\mu(t_w)}\left.\frac{\partial t}{\partial y}\right)_w \quad (9)$$

Since u is a function of A and A is a function of the temperature distribution, the velocity and temperature profiles interact mutually. Therefore the integral momentum and energy Equations (3) and (4) must be solved simultaneously. Knudsen and Katz (5) show that for $A=0$ Equation (6) represents the exact velocity profile for isothermal flow quite well. Thus for constant properties

$$\frac{u}{U}=\frac{3}{2}\frac{y}{\delta}-\frac{1}{2}\left(\frac{y}{\delta}\right)^3 \quad (10)$$

It can be shown from similarity considerations that if Equation (10) is true, then the temperature distribution for constant wall temperature is of the same form as Equation (10) where u/U is replaced by $\theta/\bar{\theta}$ and δ is re-

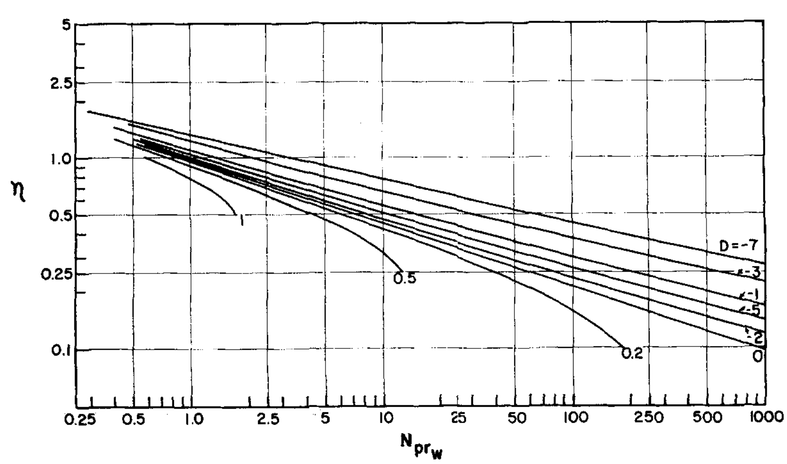

Fig. 2. Ratio of boundary-layer thicknesses vs. Prandtl number at wall temperature.

placed by δ'. Thus the temperature distribution is

$$\frac{\theta}{\bar{\theta}} = \frac{3}{2}\frac{y}{\delta'} - \frac{1}{2}\left(\frac{y}{\delta'}\right)^3 \qquad (11)$$

This can also be obtained by postulating a third-degree polynomial in y for the temperature distribution, such as Equation (5), and evaluating the four coefficients by means of the boundary conditions:

(a) $t = t_w$ at $y = 0$

(b) $t = t_\infty$ at $y = \delta'$

(c) $\frac{\partial t}{\partial y} = 0$ at $y = \delta'$

(d) $\frac{\partial^2 t}{\partial y^2} = 0$ at $y = 0$ (since k is assumed constant)

For the isothermal case Eckert's solution (2) indicates that δ is proportional to $\mu^{1/2}$, while δ' is proportional to $\mu^{1/6}$. Since the dimensionless velocity and temperature profiles in this case are the same function of δ and δ' respectively, it can be seen that the velocity profile will be affected much more by viscosity variation than the temperature profile. Further, use of analogous boundary conditions for both velocity and temperature profiles results in the variable viscosity temperature profile having the same form as that for constant viscosity. The nonisothermal temperature profile is actually different however since δ' depends upon the viscosity variation. The effect of viscosity variation upon the temperature profile does not show up directly in the boundary conditions used to evaluate coefficients in the temperature profile until $\partial^3 t/\partial y^3$ is considered. Thus it it felt that the use of the constant property form of the temperature profile in dealing with the nonisothermal problem is certainly justified.

From the temperature distribution, Equation (11), the final expression for A is obtained:

$$A = \frac{\delta\,\mu'(t_w)}{\mu(t_w)}\,\frac{3}{2}\,\frac{\bar{\theta}}{\delta'} = \frac{3}{2}\,\frac{\mu'(t_w)}{\mu(t_w)}\,\frac{\bar{\theta}}{\eta} \qquad (12)$$

Here η is the ratio of the thermal to the momentum boundary-layer thickness. From Equations (6) and (12) the effect of variable viscosity on the velocity profile is seen. Velocity profiles for various values of A are shown in Figure 1. For positive values of A (heating a liquid) the profile is seen to be flattened, while for negative values (cooling a liquid) the profiles have a point of inflection. This behavior is in accord with qualitative notions concerning velocity profiles with variable viscosity.

First, the most important problem of $\eta \leqq 1$ will be considered. This turns out to be approximately the case for $N_{Pr} \geqq 1.0$. For convenience one defines $A = D/\eta$. Combining Equations (3) and (6) and carrying out the integration one gets

$$\frac{117}{840}\frac{d}{dx}\left\{\frac{\delta}{(1-0.25A)^2}\left[1 - 0.52A + 0.0577A^2\right]\right\} = \frac{\nu_w}{U}\cdot\frac{3}{2\delta}\cdot\frac{1}{(1-0.25A)} \qquad (13)$$

The energy relation is obtained by combining Equations (4), (6), and (11) and integrating

$$\frac{d}{dx}\left\{\frac{\delta\eta^2}{1-0.25A}\left[\frac{3}{20} - \frac{D}{32} + \frac{3D\,\eta}{280}\right]\right\} = \frac{\alpha}{U}\cdot\frac{3}{2\delta\eta} \qquad (14)$$

Equations (13) and (14) are simultaneous differential equations relating δ, η, and x. For constant properties (A, $D = 0$) the momentum equation may be integrated directly, thus providing a relation between δ and x which may then be used to integrate the energy equation, so that η is found as a function of x_o, the position where heating is started.

The elimination of x from Equations (13) and (14) results in a differential equation relating δ and η. From this equation it can be shown (see Appendix) that for $x_o = 0$

$$N_{Pr_w} = \frac{1 - 0.52\dfrac{D}{\eta} + 0.0577\dfrac{D^2}{\eta^2}}{1.08\,\eta^3\left[1 - 0.208D + 0.0715D\eta\right]} \qquad (15)$$

This relates η to the variable property parameters, and it is seen that η is independent of x for $x_o = 0$. This is also true for Eckert's constant property solution. Since this is the case, Equations (13) and (14) can be integrated immediately for this case. When Equations (13) and (14) are combined (for $x_o = 0$), Equation (15) is again obtained. For constant viscosity $D = 0$ and Equation (15) reduces to the well-known cube root relation of Eckert (2).

By integrating Equation (13) and rearranging one obtains

$$\frac{\delta}{x} = \frac{4.64}{N^{1/2}_{Re_{x,w}}}\cdot\left[\frac{1-0.25A}{1-0.52A+0.0577A^2}\right]^{1/2} \qquad (16)$$

Here $N_{Re_{x,w}} = xU/\nu_w$. The friction factor is defined by

$$f = \left.\frac{2\nu_w}{U^2}\frac{\partial u}{\partial y}\right)_w \qquad (17)$$

The velocity profile, Equation (6), is combined with the expression for δ/x, Equation (16), and substituted in Equation (17) to give

$$f = \frac{0.664}{N^{1/2}_{Re_{x,w}}}\left[\frac{1-0.52A+0.0577A^2}{(1-0.25A)^3}\right]^{1/2} \qquad (18)$$

The heat transfer coefficient is defined by

$$h = \left.\frac{k}{\bar{\theta}}\frac{\partial t}{\partial y}\right)_w \qquad (19)$$

Equation (11) is used to obtain the temperature gradient at the wall. The result is combined with Equations (16) and (19) to give

$$N_{Nu_x} = \frac{0.324\,N^{1/2}_{Re_{x,w}}}{\eta}$$

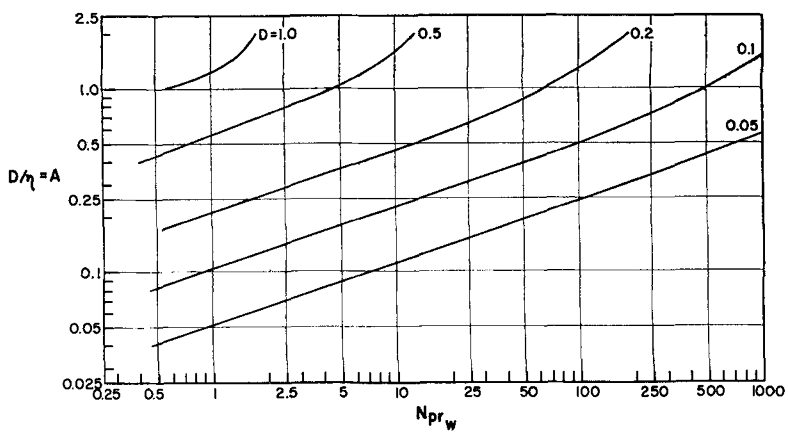

Fig. 3. Function $A = D/\eta$ vs. Prandtl number at wall temperature, D positive.

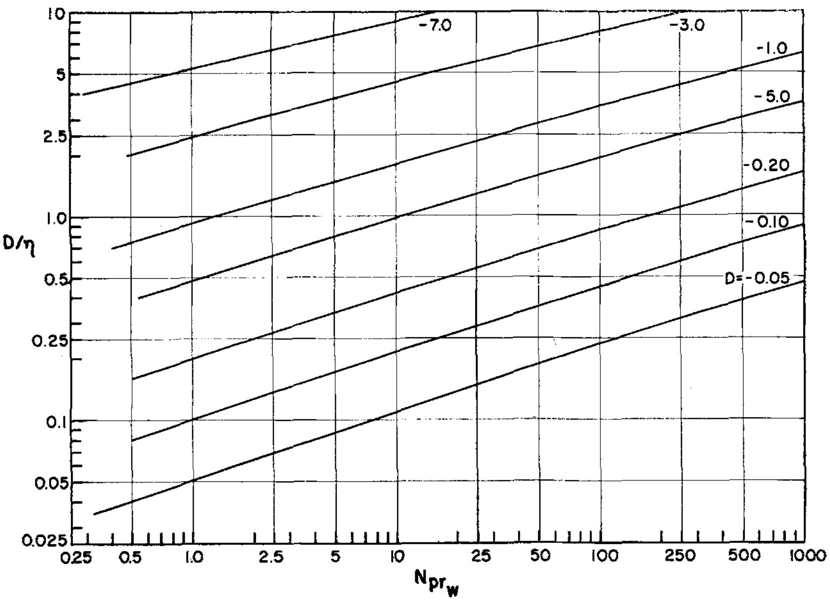

Fig. 4. Function $A = D/\eta$ vs. Prandtl number at wall temperature, D negative.

$$N_{Pr_w} = \frac{\frac{117}{840}\left[1 - 0.52\frac{D}{\eta} + 0.0577\frac{D^2}{\eta^2}\right]}{\eta\left[\frac{3}{8}\eta - \frac{(12+3D)}{32} + \frac{(6+5D)}{40\,\eta} - \frac{9D}{160\,\eta^2}\right]} \qquad (21)$$

$$\left[\frac{1 - 0.52A + 0.0577A^2}{1 - 0.25A}\right]^{1/2} \qquad (20)$$

Equations (18) and (20) reduce to Eckert's solution for A and D equal to zero.

For liquid metals, which have low Prandtl numbers, η is much larger than unity. For this case the same procedure can be used, except now the integral in Equation (4) must be broken into two parts since Equation (6) represents the velocity distribution only for $0 < y < \delta$ and $u = U$ for $\delta < y < \delta'$. With this procedure a straightforward calculation shows that for this case, when heating starts at the leading edge This relation reduces to one given by Eckert (2) for liquid metals when $D = 0$.

Equations (16) and (18) are valid for liquid metals, since the momentum Equation (13) is unchanged for $\eta > 1$. The heat transfer coefficient is again given by Equation (20), except that now η must be determined from Equation (21) rather than from Equation (15).

DISCUSSION

It can be seen from Equation (6) that the velocity profile has a singular point at $A = 4$. However before this value is reached the profile gives values of u greater than U at some points within the boundary layer. For the profile given by Equation (6) this behavior begins at $A = 2$, so that Equation (6) does not give a physically realistic velocity profile for $A > 2$. This is similar to the case of constant-property flow with a pressure gradient as discussed by Schlichting (*1*). Thus the results given here, based on Equation (6), are valid only for $A < 2$. The range of values of A for which the velocity profile is physically meaningful could be extended by assuming a higher degree polynomial for the profile, thereby making it fit more boundary conditions. The variable viscosity velocity profiles are qualitatively similar to constant viscosity velocity profiles for flow with pressure gradient (*1*).

Graphs of several of the above relations are shown. Figure 2 shows Equation (15) plotted as η vs. N_{Pr_w} for various values of D. The curves show that for positive values of D (heating a liquid) the value of η is below that predicted by the constant property equation ($D = 0$) for the same wall Prandtl number. This is to be expected because the effective viscosity in the boundary layer is greater than that at the wall, which would indicate a smaller value of η. For negative values of D (cooling a liquid) the converse argument is true. Plots of A vs. N_{Pr_w} and D for various values of D are also shown in Figures 3 and 4; from these η can be determined more accurately for a given N_{Pr_w} and D because A has a greater range of variation than η. The relation between f/f_o and A, where f_o is the isothermal friction factor evaluated at the wall temperature, is shown in Figure 5. Referring to this figure one sees that f/f_o is greater than 1.0 for $A > 0$ (heating a liquid); the effective viscosity is greater than the viscosity at the wall, and thus it would be expected that f/f_o would be greater than 1.0, which is the case. The converse argument for $A < 0$ is also applicable. Because the ratio of the actual heat transfer coefficient to the isothermal coefficient evaluated at the wall temperature depends on both N_{Pr_w} and D, it is not shown graphically here. The effect of variable viscosity on the heat transfer coefficient is considerably less than on the friction factor, since the temperature distribution is less dependent on viscosity than is the velocity distribution. However the variations in the coefficient would be expected to be slightly more than predicted here, since the effect of variable viscosity on the temperature distribution was not taken into account. Reference to Figure 6 shows the typical behavior of heat transfer coefficients.

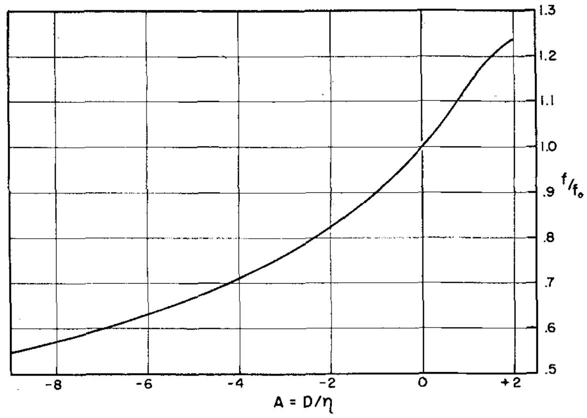

Fig. 5. Variation of friction factor ratio with function $A = D/\eta$.

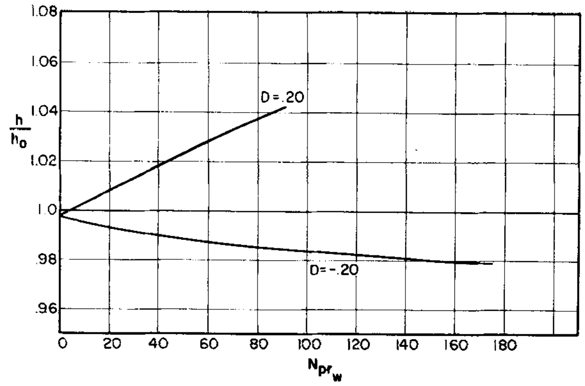

Fig. 6. Heat transfer coefficient ratio vs. Prandtl number at wall temperature.

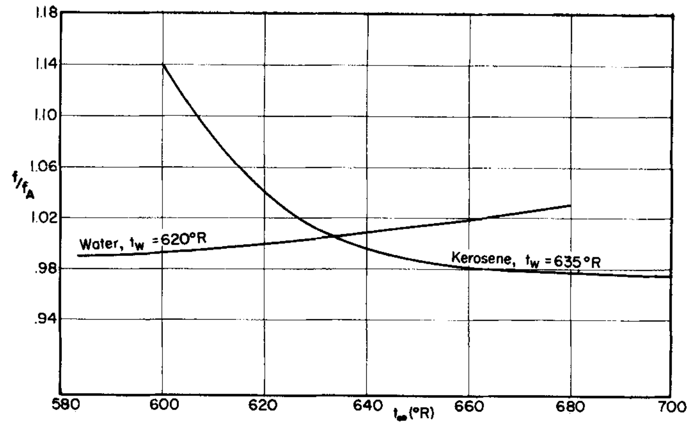

Fig. 7. Friction factor ratio vs. free stream temperature.

The method of representing the variation of viscosity with temperature does not affect the final results. Either an analytical equation or viscosity-temperature data may be used to determine the value of $\mu'(t_w)/\mu(t_w)$. When this value is known, it remains a constant in the remainder of the solution. Frequently the simple expression $(\mu/\mu_o) = (t/t_o)^n$ represents with sufficient accuracy the viscosity variation of liquids with temperature. Information of this type is available for example in Perry (6).

A comparison between f, the friction factor computed from Equation (18), and f_a, the one calculated from the isothermal relation by the arithmetic average of the wall and free stream temperature to evaluate the viscosity, is shown in Figure 7. This graph shows the ratio f/f_a vs. t_∞ for a given t_w. This demonstrates the situation for both heating and cooling of kerosene and water. The curves show that the effect of variable viscosity is greater for the fluid having a higher wall Prandtl number. This is to be expected, because the thickness of the thermal boundary layer is strongly dependent on the Prandtl number. Since the velocity profile is changed most in the vicinity of the wall owing to a variable viscosity, the temperature profile will undergo the greatest change when the thermal boundary-layer thickness is small compared with the momentum boundary-layer thickness, or for higher Prandtl numbers.

NOTATION

A = $(\delta/\mu_w)\,(\partial\mu/\partial y)_w$
C_p = heat capacity
D = A/η
f = friction factor, $2\tau_w/\rho U^2$
f_A = friction factor evaluated at $(t_w + t_\infty/2)$
f_o = friction factor evaluated at the wall temperature
h = heat transfer coefficient, $-q_w''/\overline{\theta}$
h_o = heat transfer coefficient evaluated at the wall temperature
k = thermal conductivity
$N_{Re_{x,w}}$ = xU/ν_w = Reynolds number based upon x, ν_w
N_{Pr_w} = ν_w/α = Prandtl number at the wall
N_{Nu_x} = hx/k = Nusselt number based upon x
N_{St_x} = $h/C_pU\rho$ = Stanton number based upon U
q_w'' = heat flux at the wall
t = temperature
t_w = wall temperature
t_∞ = free stream temperature
u = longitudinal velocity component
U = free stream velocity
v = transverse velocity component (normal to plate)
x = longitudinal distance variable
x_o = position where heating is begun
y = transverse distance variable (normal to plate)

Greek Letters

α = thermal diffusivity, $k/C_p\rho$
δ = momentum boundary-layer thickness
δ_o = value of δ at $x = x_o$
δ' = thermal boundary-layer thickness
η = δ'/δ
μ = viscosity
ν = kinematic viscosity
ρ = density
θ = $t - t_w$
$\overline{\theta}$ = $t_\infty - t_w$
τ_w = shear stress at the wall

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APPENDIX

Determination of η for $x_o = 0$

Letting

$$B(\eta) = \frac{\eta^2}{1 - 0.25A}\left[\frac{3}{20} - \frac{D}{32} + \frac{3D\eta}{280}\right] \tag{22}$$

and

$$E(\eta) = \frac{1 - 0.52A + 0.0577A^2}{(1 - 0.25A)^2} \tag{23}$$

one can write Equations (13) and (14) as

$$\frac{117}{840}\frac{d}{dx}\left[\delta E(\eta)\right] = \frac{\nu_w}{U}\cdot\frac{3}{2\delta}\cdot\frac{1}{(1 - 0.25A)} \tag{24}$$

$$\frac{d}{dx}\left[\delta B(\eta)\right] = \frac{\alpha}{U}\cdot\frac{3}{2\,\delta\eta} \tag{25}$$

By eliminating x from these two equations, one obtains

$$\frac{d[\delta B]}{d[\delta E]} = \frac{117}{840}\frac{(1 - 0.25A)}{N_{Pr_w}\eta} = F(\eta) \tag{26}$$

Separating variables and integrating, noting than $\eta = 0$ at $\delta = \delta_o$, one finds

$$\delta = \delta_o \exp \int_{\eta=0}^{\eta} \frac{dB - FdE}{FE - B} \tag{27}$$

Now as $\delta_o \to 0$, δ at points downstream must remain nonzero (except in the special case where $x = 0$). If this is to be true, it is seen from Equation (27) that as $\delta_o \to 0$ $(FE - B) \to 0$. Thus for the case of $\delta_o = 0$ (which is the same as $x_o = 0$)

$$EF = B \tag{28}$$

Substituting into Equation (28) the values of E, F, and B given by Equations (23), (26), and (22) respectively one finds

$$N_{Pr_w} = \frac{1 - 0.52\dfrac{D}{\eta} + 0.0577\dfrac{D^2}{\eta^2}}{1.08\,\eta^3\,[1 - 0.208D + 0.0715D\eta]}$$

This is seen to be Equation (15).